

PHYC 511
Spring 2018

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Problem Session 9

04/06/2018

(1) Problem 7.8, Jackson.

(2) Problem 7.9, Jackson.

(1) (a) The layer matrix is:

$$P(j) = \begin{bmatrix} e^{ik_j t_j} & 0 \\ 0 & e^{-ik_j t_j} \end{bmatrix} = \cos(k_j t_j) \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbb{1}} + i \sin(k_j t_j) \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\sigma_3}$$

$$= \mathbb{1} \cos(k_j t_j) + i \sigma_3 \sin(k_j t_j)$$

(b) The interface matrix $\Pi_{1 \rightarrow 2}$ is:

$$\Pi_{1 \rightarrow 2} = \begin{bmatrix} 1 & r_{21} \\ t_{21} & t_{21} \\ r_{21} & 1 \\ t_{21} & t_{21} \end{bmatrix}$$

Where:

$$r_{21} = \frac{n_2 - n_1}{n_2 + n_1} \quad , \quad t_{21} = \frac{2n_2}{n_2 + n_1}$$

$$\frac{1 - \frac{n_1}{n_2}}{1 + \frac{n_1}{n_2}} \quad , \quad \frac{2}{1 + \frac{n_1}{n_2}}$$

Thus, using $n \equiv \frac{n_1}{n_2}$, we have:

$$\Pi_{1 \rightarrow 2} = \frac{1}{2} \begin{bmatrix} 1+h & 1-h \\ 1+h & 1-h \end{bmatrix} = \frac{1}{2} (h+1) \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbb{1}} - \frac{1}{2} (h-1) \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\sigma_1}$$

$$= \boxed{\mathbb{1} \left(\frac{h+1}{2} \right) - \sigma_1 \left(\frac{h-1}{2} \right)}$$

(c) We have:

$$\begin{bmatrix} E_{trans} \\ 0 \end{bmatrix} = \Pi \begin{bmatrix} E_{inc} \\ E_{ref1} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} E_{inc} \\ E_{ref1} \end{bmatrix} \Rightarrow$$

$$\begin{cases} t_{11} E_{inc} + t_{12} E_{ref1} = E_{trans} \\ t_{21} E_{inc} + t_{22} E_{ref1} = 0 \end{cases}$$

The second equation gives:

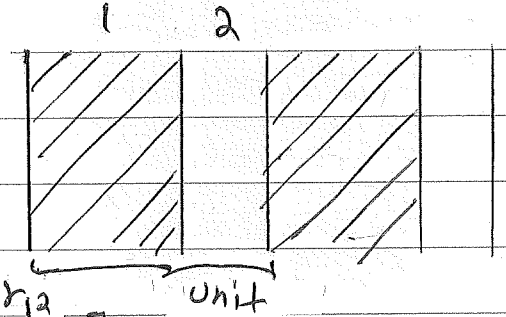
$$E_{ref1} = - \frac{t_{21}}{t_{22}} E_{inc}$$

Substituting this in the first equation yields:

$$t_{11} E_{inc} - \frac{t_{12} t_{21}}{t_{22}} E_{inc} = E_{trans} \Rightarrow E_{trans} = \frac{\det(\Pi)}{t_{22}} E_{inc}$$

(2) For each unit, the transfer matrix

is given by:



$$\Pi_{\text{unit}} = \begin{bmatrix} e^{id_2} & 0 \\ 0 & e^{-id_2} \end{bmatrix} \begin{bmatrix} 1 & r_{21} \\ t_{21} & 1 \end{bmatrix} \begin{bmatrix} e^{id_1} & 0 \\ 0 & e^{-id_1} \end{bmatrix} \begin{bmatrix} 1 & r_{12} \\ t_{12} & 1 \end{bmatrix}$$

Where:

$$r_{21} = \frac{1-h}{1+h}, \quad t_{21} = \frac{2}{1+h}, \quad t_{12} = \frac{2h}{1+h}, \quad r_{12} = \frac{h-1}{h+1} = -r_{21}$$

This results in:

$$\Pi_{\text{unit}} = \frac{(1+h)^2}{4h} (\cos d_2 \mathbb{1} + i \sin d_2 \sigma_3) (1 - r \sigma_1) (\cos d_1 \mathbb{1} + i \sin d_1 \sigma_3) (1 + r \sigma_1)$$

$r \equiv \frac{h-1}{h+1}$

After introducing ^{the} Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ and ^{using} $\delta_{ij} = i \epsilon_{ijk} \sigma_k$, we find:

$$\Pi_{\text{unit}} = \frac{(1+h)^2}{4h} [\cos d_1 \cos d_2 - \sin d_1 \sin d_2 - r^2 \cos d_1 \cos d_2 - r^2 \sin d_1 \sin d_2] \mathbb{1} - r (\sin d_1 \sin d_2 + \cos d_1 \cos d_2) \sigma_1 - 2r \sin d_1 \cos d_2 \sigma_2 + [\sin d_1 \cos d_2$$

$$+ i \sin d_2 \cos d_1 + i r^2 \sin d_1 \cos d_2 - i r^2 \sin d_2 \cos d_1] \sigma_3$$

Note that $\cos d_1 \cos d_2 - \sin d_1 \sin d_2 = \cos(d_1 + d_2)$ and $\sin d_1 \cos d_2 \pm \sin d_2 \cos d_1 = \sin(d_1 \pm d_2)$. Therefore:

$$\Pi_{\text{unit}} = \frac{1}{4n} \left\{ [(n+1)^2 \cos(d_1 + d_2) - (n-1)^2 \cos(d_1 - d_2)] \mathbb{I} + i [(n+1)^2 \sin(d_1 + d_2) + (n-1)^2 \sin(d_1 - d_2)] \sigma_1 - 2(n^2 - 1) \sin d_1 \sin d_2 \sigma_1 - 2(n^2 - 1) \sin d_1 \cos d_2 \sigma_2 \right\}$$

For the last layer, $\Pi' = \mathbb{I}_{1 \rightarrow 2} P_1 \mathbb{I}_{2 \rightarrow 1}$ since there is no air gap

to follow the last dielectric layer. Hence, $\Pi' = P_2^{-1} \Pi_{\text{unit}}$ and:

$$\Pi_{\text{stack}} = P_2^{-1} \Pi^N = P_2^{-1} (P_2 \mathbb{I}_{1 \rightarrow 2} P_1 \mathbb{I}_{2 \rightarrow 1})^N = P_2^{-1} (P_2 \mathbb{I}_{1 \rightarrow 2} P_1$$

$$\mathbb{I}_{2 \rightarrow 1})^N P_2 P_2^{-1} = (P_2^{-1} P_2 \mathbb{I}_{1 \rightarrow 2} P_1 \mathbb{I}_{2 \rightarrow 1} P_2)^N P_2^{-1} \Rightarrow$$

$$\Pi_{\text{stack}} = \underbrace{(\mathbb{I}_{1 \rightarrow 2} P_1 \mathbb{I}_{2 \rightarrow 1} P_2)^N}_{\Pi \text{ in the book}} P_2^{-1} = \Pi^N (\cos d_2 \mathbb{I} - i \sin d_2 \sigma_3)$$

(has to do with its choice of unit)

(b) For $d_1 = d_2 = \frac{\pi}{2}$, we have:
 $\Rightarrow \frac{\lambda}{4}$ wavelength thickness

$$\Pi = \frac{1}{4n} \left[-(n+1)^2 - (n-1)^2 \right] \mathbb{I} + \frac{1}{2n} (n^2 - 1) \sigma_1 = -\frac{(n^2 + 1)}{2n} \mathbb{I} - \frac{n^2 - 1}{2n} \sigma_1$$

Note that:

$$e^{\lambda \sigma_1} = \left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots\right) \mathbb{1} + \left(\lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \dots\right) \sigma_1 = \cosh \lambda \mathbb{1} + \sinh \lambda \sigma_1$$

For $\lambda = \ln n$, we have:

$$\cosh \lambda = \frac{e^\lambda + e^{-\lambda}}{2} = \frac{n^{\frac{1}{2}} + 1}{2}, \quad \sinh \lambda = \frac{e^\lambda - e^{-\lambda}}{2} = \frac{n^{\frac{1}{2}} - 1}{2}$$

Thus:

$$\mathbb{\Pi} = -e^{-\lambda \sigma_1}$$

And:

$$\det(\mathbb{\Pi}_{stack}) = \det(\mathbb{\Pi}^N P_2^{-1}) = (\det \mathbb{\Pi})^N (\det P_2)^{-1} = (-1)^{2N} e^{-\ln n \text{Tr} \sigma_1}$$

Here, we have used $\det(e^A) = e^{\text{Tr} A}$. Since $\text{Tr} \sigma_1 = 0$, we have:

$$\det(\mathbb{\Pi}_{stack}) = 1$$

On the other hand:

$$(\mathbb{\Pi}_{stack})_{22} = (-1)^N (e^{-N \ln n \sigma_1})_{22} = (-1)^N \cosh(N \ln n)$$

Therefore:

$$\left| \frac{E_{trans}}{E_{inc}} \right|^2 = \frac{1}{\cosh^2(N \ln n)} = \frac{1}{(n^{\frac{N}{2}} + n^{-\frac{N}{2}})^2} \xrightarrow{N \rightarrow \infty} 4n^{-2N} = 4e^{-N \ln n^2}$$